and intuitively natural approach to mathematical analysis that he named nonstandard analysis. Nonstandard analysis is one of the few innovations in logic that were entirely the work of a single individual.

Not long before his death, Robinson collaborated with the number theorist Peter Roquette to apply model-theoretic methods in number theory. This work gave a first hint of the deep interactions between model theory and diophantine geometry that came to light in the 1990s, sadly too late for Robinson to contribute. In fact, Robinson died before he could take on board the stability theory pioneered by Michael Morley and Saharon Shelah, though his students, Greg Cherlin and Carol Wood, did contribute to this field, bringing with them Robinson's lifelong eagerness to apply model theory to algebra, algebraic geometry, and mathematics in general.

Though unable himself to believe in any kind of existence for infinite totalities, he strongly defended the right of mathematicians to proceed as if such totalities exist. His discussion (Robinson 1965) of mathematical and epistemological considerations that favor one or another of the traditional views in philosophy of mathematics is thoughtful but seems not to reveal a thoroughly worked out position. His anti-Platonistic attitude may have helped him to create nonstandard analysis by allowing him to be relaxed about what the "real" real numbers are.

In Robinson's *Selected Papers* (1979), the bibliography lists ten books, more than a hundred papers, and a film. One in seven of his papers are in wing theory and aeronautics.

See also Infinitesimals; Model Theory; Tarski, Alfred.

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SET THEORY SINCE GÖDEL See *Set Theory*

THE PROLIFERATION OF NONCLASSICAL LOGICS. The twentieth century, and especially its second half, was marked by a fairly spectacular proliferation of what are sometimes called nonclassical logics. To understand this, one needs to see the matter in its historical context. There have been three great periods in the history of European logic: ancient Greece, medieval Europe, and, starting toward the end of the nineteenth century, the current period. Each period has been marked by the production of novel theories of the nature and extent of logical validity. Thus, in the ancient period, Aristotle, the Megarian, and the Stoic logicians offered different accounts of validity, the conditional, and modality. The medieval period tried to reconcile some of the differences in their heritage, and in the process produced numerous different accounts of the nature of the connectives, consequence, and supposition. Not surprisingly, in both periods there was active and lively debate concerning the theories that were produced.

The periods between the great periods were characterized not just by a lack of interest in logic, but by a forgetting of much of the significant prior developments. In particular, all that remained of logic in about the middle of the nineteenth century—so-called traditional logic—was a somewhat bowdlerized form of the theory of the syllogism and some of its medieval accompaniments. It was at this time that mathematical logic came into existence. It was mathematical in two senses. The first is that the logicians who produced it were interested in the analysis of the reasoning of the mathematics of their time (and its foundations). The second is that they applied mathematical techniques to the subject in a novel way, such as those of abstract algebraic, set theory, and combinatorics.

Out of this, principally at the hands of Gottlob Frege and Bertrand Russell, developed a novel theory of logic. This was streamlined, organized, and simplified by a number of logicians in the first part of the twentieth century—notably, David Hilbert, Alfred Tarski, and Gerhard Gentzen. The result was an account of inference that was so much more powerful than traditional logic that is soon superseded it as the standard canon. This is so-called classical logic.

It had hardly appeared, however, before some logicians realized that a number of assumptions that were packed into it were contentious—especially once one goes beyond the kind of mathematical reasoning out of which classical logic arose. One of these was the principle of bivalence: that every (declarative) sentence is either true or false. In the 1920s the first many-valued logics were produced by Jan Łukasiewicz, Emil Post, Tarski, and others. In many-valued logics, sentences can be assumed to be neither true nor false, both true and false, have an infinity of degrees of truth, and so on.

Another assumption that is packed into classical logic is truth-functionality: that the truth value of a compound sentence is a function of the truth values of its parts. This is obviously not true of modal notions, and in the 1920s Clarence Irving Lewis presented in axiomatic form the first modern systems of modal logic. Modal logic was given an enormous boost with the discovery of world-semantics by, in particular, Saul Kripke in the 1960s. This allowed for the production of logics for other non-truth-functional notions (so called intentional logics), such as tense-operators (by Arthur Prior), epistemic and doxastic notions (by Jaako Hintikka), and deontic notions (by Henrik von Wright).

Another early critique of classical logic was provided by mathematical intuitionists, such as Luitzen Brouwer and Arend Heyting, who, driven by the view that existence should not be asserted unless people can construct the object in question, produced a system of formal logic in which a number of propositional and quantifier inferences that are valid in classical logic fail.

In the second half of the century, various critics of classical logic attacked the account of the (material) conditional it employs (as had Lewis). This produced the relevant logics of Alan Anderson and Nuel Belnap, and the conditional logics of Robert Stalnaker and David Lewis. These logics both have world-semantics. The world semantics for relevant logics were produced by, in particular, Richard Routley (later Sylvan) and Robert Meyer. The central feature of such semantics (it can be seen in retrospect) is the deployment of the notion of an impossible world.

The principle of inference of classical logic that everything entails a contradiction came under attack in its own right by logicians in the same period, including Stanisław, Jaśkowski, Newton da Costa, and Graham Priest. This produced a number of paraconsistent logics, which may be many-valued, modal, relevant, or of other kinds.

The development of nonclassical logics received further momentum from the advent of computer science and information technology after the 1960s. This produced new constructivist systems (such as linear logic), intentional logics (such as dynamic logic), and paraconsistent logics (such as various resolution systems). Research in Artificial Intelligence has also produced new epistemic logics, as well as the whole new area of formal non-monotonic (i.e., non-deductive) inference.

Thus, at the start of the twenty-first century there is a wide range of logics embodying different metaphysical presuppositions and potential applications. What to make of this is another matter. Perhaps most obvious is that the revolution in logic that occurred around the turn of the twentieth century was not so much the production of a novel logical theory—important though this was. It was instead the deployment of mathematical techniques to logic in a novel way. This allowed the development of classical logic, but the techniques were so powerful and versatile that they could be used to produce many other logics as well.

Which of all these logics is right, and, indeed, the meaning of that question, are matters to be determined only by detailed philosophical argument. Such arguments have been much part of the philosophical landscape since about the middle of the twentieth century. Indeed, the twenty-first century is seeing disputes in philosophical logic of a depth and acuity not seen since medieval logic. Whatever their outcome, the presence of the multitude of logical systems serves to remind that logic is not a set of received truths, but a discipline in which competing theories concerning validity vie with each other. The case for each theory—including a received theory—has to be investigated on its merits.

See also Aristotle; Brouwer, Luitzen Egbertus Jan; Conditionals; Frege, Gottlob; Hintikka, Jaako; Hilbert, David; Intuitionism and Intuitionistic Logic; Kripke, Saul; Lewis, Clarence Irving; Lewis, David; Logic, Non-Classical; Łukasiewicz, Jan; Many-Valued Logics; Megarians; Modal Logic; Non-Monotonic Logic; Paraconsistent Logics; Prior, Arthur Norman; Relevance (Relevant) Logics; Russell, Bertrand Arthur William; Stoicism; Tarski, Alfred; Wright, Georg Henrik von.

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KRIPKE AND KRIPKE MODELS. See *Kripke, Saul*

FRIEDMAN AND REVERSE MATHEMATICS. During the second half of the twentieth century, many mathematicians lost interest in the foundations of mathematics. One of the reasons for this decline was an increasingly popular view that general set theory and Gödel-style incompleteness and independence results do not have much effect on mathematics as it is actually practiced. That is, as long as mathematicians study relatively concrete mathematical objects, they can avoid all foundational issues by appealing to a vague hybrid of philosophical positions including Platonism, formalism, and sometimes even social constructivism. Harvey Friedman (born 1948) has continually fought this trend, and in 1984 he received the National Science Foundation's Alan T. Waterman Award for his work on revitalizing the foundations of mathematics.

One of Friedman's methods of illustrating the importance of foundational issues is to isolate pieces of mathematics that either display the incompleteness phenomenon or require substantial set theoretic assumptions and which most mathematicians would agree fall within the scope of the central areas of mathematics. For example, he has created numerous algebraic and geometric systems that make no explicit reference to logic but which, under a suitable coding, contain a logical system to which Gödel's incompleteness theorems apply. Furthermore, these systems look similar to many systems used by mathematicians in their everyday work. Friedman uses these examples to argue that incompleteness cannot be dismissed as a phenomenon that occurs only in overly general foundational frameworks contrived by logicians and philosophers.

Friedman has also done a large amount of work concerning the necessary use of seemingly esoteric parts of Zermelo-Frankel set theory and its extensions. He has found theorems concerning concrete objects in mathematics that require the use of uncountably many iterations of the power set axiom and others that require the use of large cardinal axioms. These investigations have culminated in what Friedman calls Boolean relation theory.

In his 1974 address to the International Congress of Mathematicians, Friedman started the field of reverse mathematics by suggesting a three-step method for measuring the complexity of the set theoretic axioms required to prove any given theorem T. First, formalize the theorem T in some version of set theory. (Typically a formal system called second order arithmetic is used.) Second, find a collection of set theoretic axioms S which suffices to prove T. Third, prove the axioms in S from the theorem T (while working in a suitably weak base theory). If the third step is successful, then the equivalence between S and T shows that S is the weakest collection of axioms which suffices to prove T. If the third step fails, then the second step must be repeated until a proof of T is found using only axioms that can be proved from T. Because the third step involves proving axioms from theorems as opposed to the usual action of proving theorems from axioms, this type of analysis is now called reverse mathematics. It is frequently possible to draw a number of foundational conclusions concerning a theorem T once the equivalent collection S of set theoretic axioms has been isolated.

See also Gödel's Incompleteness Theorems; Mathematics, Foundations of; Platonism and the Platonic Tradition; Reverse Mathematics.

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LOGIC, MATHEMATICAL

See *Logic, History of*

LOGIC, MODAL

See *Modal Logic*