Fuzzy Relevant Logic

Dedicated to Newton da Costa on the occasion of his 70th birthday.

Abstract

This paper raises the question of what a logic should be like if it is both fuzzy and relevant. Two strategies are considered for answering the question. In the first, standard world-semantics for relevant logics are "fuzzified". In the second, algebraic semantics for relevant logics are simply reinterpreted, showing that we can think of standard relevant logics as already fuzzy. The two strategies deliver a number of logics with different properties, especially concerning the conditional.

1 Relevance, Vagueness and Paraconsistency

The study of paraconsistent logic is now about 50 years old. A major pioneer of the subject, Newton da Costa, articulated many paraconsistent logics, showing the way to this rich and important field. There are now very many different kinds of paraconsistent logic; and they have been suggested with very many different applications in mind.[1] Two such applications, which will concern us in this paper, are vagueness and relevance.

Let us start with relevance. The thought that there must be some connection between the antecedent and consequent of a true conditional is an ancient and very natural one. A (propositional) relevant logic is one which respects this intuition in the following form: whenever $A \to B$ is a logical truth, A and B share a propositional parameter. In particular, then, $\nvDash (p \wedge \neg p) \to q$. Strictly speaking, relevant logics need not be paraconsistent. For example, Ackermann's system Π' was relevant. However, one of its primitive rules was

[1] For a survey, see Priest [6].

the disjunctive syllogism, and (interpreting this as a rule of derivability) it quickly gives $p, \neg p \vDash q$. Still, the same kind of intuition that rebels against the logical truth of conditionals of the form $(p \wedge \neg p) \rightarrow q$ rebels against the validity of inferences of the form $p, \neg p \vdash q$. It is not, therefore, surprising that most relevant logics are paraconsistent as well. Indeed, Ackermann's Π' was reworked by Anderson and Belnap into their favourite relevant logic, E, which is paraconsistent.

A quite different motivation for paraconsistency comes from the notion of vagueness. Most of the normal predicates we operate with are vague. Specifically, there appear to be transition areas where their applicability fades out or fades in. Thus, for example, as a child grows into an adult, there would seem to be a transitional period around adolescence, where they would seem to be as much child as adult, or as little adult as child. What status does the claim that the person in question is a child have during this period, when they are symmetrically poised between childhood and adultery? The natural answers are also symmetric: neither a child nor not a child; both a child and not a child. Common sense seems to be comfortable with both possibilities, though most logicians have taken only the second seriously. If one does, though, one clearly needs a paraconsistent logic. For during the transition period it is not true, for example, that the person is a chicken. Hence, this cannot follow from the contradictory characterisation.

Actually, there seems to be more to vagueness than so far indicated. Any simple semantic dichotomy or trichotomy appears to be inadequate to characterise vagueness. As the child grows up, there seems to be no precise line between being a child and not being a child; or between being a child and neither being nor not being a child; or between being a child and both being and not being a child. In virtue of this, it is very natural to suppose that there are degrees of truth. A standard way of implementing this idea is by representing truth values as real numbers in the closed interval $[0, 1]$. If one gives a natural semantics for negation then, again, a paraconsistent logic is obtained. For it is easy to arrange for the value of $p \wedge \neg p$ to be greater than that of q, making the inference from $p \wedge \neg p$ to q invalid. We will come back to the details of this in a moment.

All of this background will be familiar to most paraconsistent logicians. Now to the main issue I want to raise. Intuitively, at least, there is nothing incompatible about relevance and vagueness—quite the opposite: the conditional 'if John is a child and not a child then he is a chicken', seems intuitively quite rebarbative, even though the predicates in question are vague.

Yet, though the studies of relevance and of degrees of truth have both given rise to paraconsistent logics, relevant logics and fuzzy logics are currently quite distinct. Standard relevant logics countenance only the truth values *true* and *false*—though sometimes they may be allowed to occur in combination. And standard fuzzy logics are certainly not relevant. How, then, to put these two ideas together? What should a fuzzy relevant logic be like? Surprisingly, that question seems scarcely to have been raised.[2] It is the issue I want to address in the rest of this paper. I will consider two strategies for producing a fuzzy relevant logic: "fuzzification" and reinterpretation. The considerations are purely technical: I shall not discuss the philosophical adequacy of any of the logics concerned here.

2 Strategy 1: Fuzzification

2.1 Fuzzy Logic

To see how fuzzification works, let us start with a clean statement of a fuzzy logic. There are many such logics. Standard ones differ as to how they give the truth conditions of connectives. Here, I will employ the connectives that are probably most familiar to philosophers, the Łukasiewicz truth conditions.[3] It should be clear that fuzzification could be performed in exactly the same way with others.

Truth values are represented by the closed interval, $[0, 1]$. If ν is an assignment of truth values to propositional parameters, this is extended to other formulas by the following clauses:

$\nu(\neg A) = 1 - \nu(A)$

[2]There have been some near misses. Peña's fuzzy logic of [4] is an extension of the relevant system E, but it is not a relevant logic. Closer, Slaney's logic F of [8] is clearly in the same family as standard relevant logics, but it, too, is not relevant. Sylvan and Hyde [10] argue that a relevant logic without *modus ponens* is a suitable logic for vagueness, but the logic is not fuzzy, having just the usual two truth values. In a sense, Boričić [1] fuzzifies possible-world semantics for intuitionist and stronger logics. The construction adds operators like 'Is true to at least degree r' to the language itself; but it leaves the semantics two-valued. The same techniques can be applied to other possible-world semantics, including those for relevant logics.

[3]Technically, standard fuzzy logics turn around the notion of a *t-norm*, which determines the behaviour of the conditional (and the conjunction of which it is the residuum). Apart from the Łukasiewicz t-norm, the best known are those of Gödel and Product logics. For details, see Hájek [2], esp. chs. 2-4.

$\nu(A \wedge B) = Min(\nu(A), \nu(B))$

$\nu(A \vee B) = Max(\nu(A), \nu(B))$

$\nu(A \rightarrow B) = \nu(A) \ominus \nu(B)$

where:[4]

$$\begin{array}{lll} a \ominus b & = 1 & \text{if } a \leq b \\ a \ominus b & = 1 - (a - b) & \text{if } a > b \end{array}$$

In many-valued logics validity is defined in terms of the preservation of designated values. In standard fuzzy logics, the set of designated values is taken to be $\{1\}$. We will be a little more general here. It is natural to think of the designated values as the values of those things that are true enough to be acceptable, and taking 1 to be the only such value seems a little overzealous. Technically, the set of designated values could be any subset of $[0, 1]$. However, if we are thinking of designated values in the way just explained, it is natural to require the designated values to be closed upwards. Hence, if $0 \leq \varepsilon \leq 1$, any set of the form $[\varepsilon, 1]$ is a possible set of designated values, defining a corresponding notion of validity. Thus, we have $\Sigma \models_\varepsilon B$ iff $\forall \nu$(if $\forall A \in \Sigma, \nu(A) \geq \varepsilon$ then $\nu(B) \geq \varepsilon$).[5]

ε is the lower bound of those degrees of truth that are acceptable; and it is plausible to suppose that this is a contextual matter. In some contexts (for example, choosing a safe drug where someone's life is at stake), one would require a higher degree than others (for example, choosing a coloured paint where one is decorating a house). Hence, it makes sense to abstract from context, and define an absolutely valid inference as one that is valid, no matter what ε is; that is, $\Sigma \models B$ iff $\forall \varepsilon \, \Sigma \models_\varepsilon B$. It is not difficult to establish that this is equivalent to the following definition. $\Sigma \models B$ iff:

$$\forall \nu(Glb(\nu[\Sigma]) \leq \nu(B))$$

where $\nu[\Sigma] = \{\nu(A) \colon A \in \Sigma\}$, and $Glb(X)$ is the greatest lower bound of X (between 0 and 1).[6]

[4] In Gödel and Product logics, the definiens in the second clause is replaced by b, and b/a, respectively.

[5] Alternatively, we could take the designated values to be the half-open interval, $(\varepsilon, 1]$, and replace the '$\leq$'s with '$<$'s.

[6] *Proof*: Suppose that $\forall \nu(Glb(\nu[\Sigma]) \leq \nu(B))$. Now, suppose, for fixed ε, that $\forall A \in \Sigma, \nu(A) \geq \varepsilon$. Then $Glb(\nu[\Sigma]) \geq \varepsilon$. Hence $\Sigma \models_\varepsilon B$. Thus, $\Sigma \models B$. Conversely, suppose that for some ν, $Glb(\nu[\Sigma]) = \varepsilon > \nu(B)$. Then for all $A \in \Sigma$, $\nu(A) \geq \varepsilon > \nu(B)$. Thus, $\Sigma \not\models_\varepsilon B$, and hence, $\Sigma \not\models B$.

In particular, if $\Sigma = \{A_1, ...A_n\}$, then $Glb(\nu[\Sigma]) = Min(\nu(A_1), ..., \nu(A_1))$, so $\Sigma \models A$ iff $\nu(A_1 \wedge ... \wedge A_n) \leq \nu(B)$ for all ν; and if Σ is empty, then $Glb(\nu[\Sigma]) = 1$. Thus, $\models B$ (i.e., $\phi \models B$) iff $\forall \nu\, \nu(B) = 1$.[7]

One of the most distinctive features of fuzzy logic, thus formulated, is the failure of *modus ponens*—as one might expect, given its role in sorites paradoxes. To see this, set $\nu(p) = 0.75$, $\nu(q) = 0.5$. Then $\nu(p \to q) = 0.75$. Hence $\nu(p \wedge (p \to q)) = 0.75 > \nu(q)$. Note also that for any ν, $\nu(p \to (q \to q)) = 1$. Hence, $\models p \to (q \to q)$. As I observed in the introduction, Łukasiewicz' fuzzy logic is not a relevant logic.[8] It is also known not to be compact.[9] It follows that it has no sound and complete proof theory. It does have a proof theory sound and complete with respect to finite sets of premises, though.[10]

2.2 Fuzzy Modal Logic

Before we turn to relevant logic, let us see how fuzzification works in the slightly simpler case of modal logic. Specifically, let us see how to fuzzify the simplest normal modal logic, K. As is well known, an interpretation for K is a structure $\langle W, R, \nu \rangle$, where W is a set of worlds, R is an arbitrary binary relation on W, and for every $w \in W$, and propositional parameter, p, $\nu(w, p) \in \{0, 1\}$. (I will write $\nu(w, A)$ as $\nu_w(A)$.) The truth conditions for a standard set of connectives are as follows. For all $w \in W$:

$\nu_w(\neg A) = 1$ iff $\nu_w(A) = 0$

$\nu_w(A \wedge B) = 1$ iff $\nu_w(A) = \nu_w(B) = 1$

[7] So the logical truths of this logic coincide with the fuzzy logic in which 1 is taken to be the sole designated value.

[8] If one takes 1 to be the only designated value, it is not even paraconsistent, since $\alpha, \neg\alpha \models_1 \beta$.

[9] To see this, define $A \oplus B$ as $\neg A \to B$. Then it is not difficult to check that $\nu(A \oplus B) = Min(\nu(A) + \nu(B), 1)$. Let $1A$ be A, and $(n+1)A$ be $A \oplus nA$. Then if $\nu(A) > 0$, the sequence: $\nu(1A), \nu(2A), \nu(3A)...$ eventually takes the value 1. Let $\Sigma = \{p, \neg 1p, \neg 2p, \neg 3p, ...\}$. Then it follows that for any ν, there is a $B \in \Sigma$ such that $\nu(B) = 0$. Hence, $\Sigma \models q$. But there is no finite $\Sigma' \subseteq \Sigma$ such that $\Sigma' \models q$. For let m be the greatest n such that $nA \in \Sigma'$. Let $\nu(p) = 1/(m+1)$. Then it is easy to check that if $A \in \Sigma'$, $\nu(A) \geq 1/(m+1)$. Now let $\nu(q) = 0$, to see that $\Sigma' \not\models q$.

[10] Observe that $\{A_1, A_2, ..., A_n\} \models B$ iff $\models_1 (A_1 \wedge A_2 \wedge ... \wedge A_n) \to B$. Hence, an inference with a finite set of premises is valid iff the corresponding conditional is logically true in Łukasiewicz' continuum-valued logic with designated value 1. This is well known to be axiomatizable. See, for example, Hájek [2], ch. 3.

$\nu_w(A \supset B) = 1$ iff $\nu_w(A) = 0$ or $\nu_w(B) = 1$

$\nu_w(\Box A) = 1$ iff for all w' such that wRw', $\nu_{w'}(A) = 1$

A little thought shows that these may be expressed equivalently as:

$\nu_w(\neg A) = 1 - \nu_w(A)$

$\nu_w(A \wedge B) = Min(\nu_w(A), \nu_w(B))$

$\nu_w(A \supset B) = \nu_w(A) \ominus \nu_w(B)$

$\nu_w(\Box A) = Glb\{\nu_{w'}(A); wRw'\}$

In particular, for the modal operator: if $\nu_{w'}(A) = 1$ for all w' such that wRw', $Glb\{\nu_{w'}(A); wRw'\} = 1$; and if $\nu_{w'}(A) = 0$ for some w' such that wRw', $Glb\{\nu_{w'}(A); wRw'\} = 0$.

The standard definition of logical consequence for K is:

$\Sigma \models A$ iff for every $\langle W, R, \nu\rangle$ and $w \in W$, if $\nu_w(B) = 1$ for all $B \in \Sigma$, $\nu_w(A) = 1$

Again, a little thought shows that this may be expressed equivalently as a simple generalisation of the fuzzy definition of the previous section:

$\Sigma \models A$ iff for every $\langle W, R, \nu\rangle$ and $w \in W$, $Glb(\nu_w[\Sigma]) \leq \nu_w(A)$

Fuzzifying K is now completely routine. We simply take the above account, where the truth conditions and definition of validity are expressed in the equivalent terms, and replace $\{0, 1\}$ by $[0, 1]$. Let us call the result FK (*fuzzy* K).

The relationship between K and FK is not difficult to establish. For a start, any K counter-model is a special case of an FK counter-model (where everything takes the value 1 or 0). Hence if $\Sigma \models_{FK} A$, then $\Sigma \models_K A$. The converse is not true, however. For *modus ponens* in valid in K, but invalid in FK. (Just consider the one-world model corresponding to the counter-model of the last section.)

As should be clear, other modal logics with world semantics can be fuzzified in exactly the same way. Thus, for example, fuzzified $S4$ is the same as FK, except that R is required to be reflexive and transitive.[11] Similarly, we

[11]Fuzzy versions of some modal logics, and in particular $S5$, are already known. See Hájek [2], 8.3, for discussion and references.

can fuzzify intuitionist logic by fuzzifying its world semantics. However, I will not go into any of this here, since this is not a paper about modal logic. The preceding material was just to illustrate the basic idea of fuzzification, which we will now apply to relevant logic.

2.3 Fuzzifying Relevant World Semantics

Perhaps the simplest and most natural semantics for relevant logics are world-semantics. Leave negation aside for a moment. A very simple world semantics for a positive relevant logic is a structure $\langle N, W, \nu\rangle$, where W is a set of worlds; N is a subset of W, the normal worlds—the rest being non-normal; and ν is an evaluation function that assigns a truth value, 0 or 1, to every propositional parameter at every world, and to every conditional at every non-normal world. Truth values are then assigned to other formulas by the recursive clauses:

For $w \in W$: $\nu_w(A \wedge B) = 1$ iff $\nu_w(A) = 1$ and $\nu_w(B) = 1$

For $w \in W$: $\nu_w(A \vee B) = 1$ iff $\nu_w(A) = 1$ or $\nu_w(B) = 1$

For $w \in N$: $\nu_w(A \rightarrow B) = 1$ iff for all $w' \in W$ such that $\nu_{w'}(A) = 1$, $\nu_{w'}(B) = 1$.

(The truth values of conditionals at non-normal worlds are already taken care of by ν.) The truth conditions may be expressed equivalently thus:

For all $w \in W$: $\nu_w(A \wedge B) = Min(\nu_w(A), \nu_w(B))$

For all $w \in W$: $\nu_w(A \vee B) = Max(\nu_w(A), \nu_w(B))$

For all $w \in N$: $\nu_w(A \rightarrow B) = Glb\{\nu_{w'}(A) \ominus \nu_{w'}(B);\ w' \in W\}$

Validity is defined as truth preservation at all *normal* worlds of all interpretations:

$\Sigma \models A$ iff for all $\langle W, N, R, \nu\rangle$ and $w \in N$, if $\nu_w(B) = 1$ for all $B \in \Sigma$, then $\nu_w(A) = 1$

Or equivalently:

$\Sigma \models A$ iff for all $\langle W, N, R, \nu\rangle$ and $w \in N$, $Glb\{\nu_w(B);\ B \in \Sigma\} \leq \nu_w(A)$

These semantics give the positive relevant logic H^+. One of its characteristic features is that it has no entailments that involve conditionals essentially. That is, any logical truth of the form $A \to B$ is a substitution instance of one of the form $A' \to B'$, where A' and B' contain no occurrences of $\to$. H^+ is, at any rate, a relevant logic.[12]

An axiom system for H^+ is as follows.

A1 $A \to A$

A2 $(A \wedge B) \to A \qquad (A \wedge B) \to B$

A3 $A \to (A \vee B) \qquad B \to (A \vee B)$

A4 $A \wedge (B \vee C) \to ((A \wedge B) \vee (A \wedge C))$

R1 $A, A \to B \vdash B$

R2 $A, B \vdash A \wedge B$

R3 $A \to B, B \to C \vdash A \to C$

R4 $A \to B, A \to C \vdash A \to (B \wedge C)$

R5 $A \to C, B \to C \vdash (A \vee B) \to C$

The axiom system is weakly complete (i.e., complete for the empty set of premises). For strong completeness (i.e., completeness for arbitrary sets of premises), the disjunctive forms of the rules of inference have to be added as well. (The disjunctive form of R1 is: $A \vee C, (A \to B) \vee C \vdash B \vee C$. The others are similar.)

To transform these semantics into a fuzzy logic, we simply replace the set of truth values $\{0, 1\}$ with the closed interval $[0, 1]$, as we did for K, taking the truth conditions and definition of validity in their equivalent forms. Let us call this system FH^+.

As with K, all two-valued interpretations are fuzzy interpretations. It follows that if $\Sigma \vDash_{FH^+} B$ then $\Sigma \vDash_{H^+} B$. In particular, then, FH^+ is a relevant logic. Moreover, again as for K, the implication does not go in the opposite direction, since *modus ponens* fails in FK^+. (Take an interpretation with one world, w, which is normal, where $\nu_w(p) = 0.75$, and $\nu_w(q) = 0.5$.)

[12]For further details concerning this, and all the other facts about relevant logic referred to in this paper, see Priest [6].

This time, however, we can say a little more. It is easy to check that each of the axioms of H^+ is logically valid in FH^+, and that each of the rules of inference (including the disjunctive forms) preserves logical validity. It follows that if $\models_{H^+} B$ then $\models_{FH^+} B$. Thus, H^+ and FH^+ have the same logical truths.[13] Whether there is a complete proof theory for FH^+, or even just for finite sets of premises, is presently an open question.[14]

2.4 Negation

So much for a basic relevant fuzzy logic. How should this be extended to include negation? There are standardly two treatments of negation that may be joined to the semantics of H^+ to give a full relevant logic.

The first of these relaxes the condition that ν be a function. Instead, it is a relation, relating each formula to some, all, or none, of the truth values, 1 and 0. One can pursue this strategy in the fuzzy case, too, but it is problematic. Let $\nu_w[A] \subseteq [0,1]$ be the set of values to which A is related by ν_w. What now are the appropriate truth conditions for the connectives, and definition of validity? Perhaps the most natural truth conditions are the combinatorial ones. E.g.:

$$x \in \nu_w[A \wedge B] \text{ iff } \exists y \in \nu_w[A], \exists z \in \nu_w[B], x = Min(y,z)$$

Thus, the values of $A \wedge B$ are the values that one can get by combining all the values of A and B in the usual way. And $\Sigma \models B$ iff for all $\langle W, N, \nu \rangle$ and $w \in N$:

$$Glb\{Glb(\nu_w[A]);\ A \in \Sigma\} \ \leq Glb(\nu_w[B])$$

This certainly makes sense, but it is not the generalisation of the two valued case that one would expect. For if, say, $\nu_w[A] = \{0\}$ and $\nu_w[B] = \phi$ then $\nu_w[A \wedge B] = \phi$. Consequently, $A \wedge B \not\models A$. (In the non-fuzzy case, $A \wedge B \models A$, since if $\nu_w[A] = \{0\}$ and $\nu_w[B] = \phi$ then $\nu_w[A \wedge B] = \{0\}$. One false conjunct is sufficient to make the conjunction false.) Insisting that for all A and w, $\nu_w[A]$ be non-empty does not help, either. For then, for all w, $Glb(\nu_w[p \wedge \neg p]) \leq 0.5$ and $Glb(\nu_w[q \vee \neg q]) \geq 0.5$. Hence, $\models (p \wedge \neg p) \to (q \vee \neg q)$.

[13] This is not the same for K: $A \vee \neg A$ is logically valid in K, but not FK.

[14] Another option for creating a fuzzy relevant logic here (and in what follows) is to retain the definition of logical validity in its original form. That is, validity is defined in terms of preservation of the value 1 (at all normal worlds). This gives a stronger logic with the same set of logical truths. In particular, *modus ponens* is valid for this logic.

Whether there are other ways of generalising the relational case, with less untoward consequences, I do not know. But the other way of handling negation in relevant logic is more straightforward. In this, an interpretation is augmented with an operator on worlds, $*$ (the Routley star), such that for all $w \in W$, $w^{**} = w$. The truth conditions of negation are then:

$\nu_w(\neg A) = 1$ iff $\nu_{w^*}(A) = 0$

Or equivalently:

$\nu_w(\neg A) = 1 - \nu_{w^*}(A)$

Adding this machinery to H^+ gives the logic H.

To obtain an axiom system for H, we add the following rules and axioms to those for H^+:

A5 $A \leftrightarrow \neg\neg A$

R6 $A \rightarrow B \vdash \neg B \rightarrow \neg A$

where $A \leftrightarrow B$ is $(A \rightarrow B) \wedge (B \rightarrow A)$.

The generalisation of this to the fuzzy case is obvious. Formulate the semantics in the appropriate terms, and simply replace {0,1} with [0,1]. Call the system produced FH. The relation between H and FH is the same as that between H^+ and FH^+, and for exactly the same reason as before. The logical truths of the two are the same; but for deducibility, FH is a proper sublogic of H. Again, whether there is a complete proof theory for FH^+, or even just for finite sets of premises, is presently an open question.[15]
There remains, also, the question of what $*$ means, and why it should poke its nose into the truth conditions of negation. This is an unresolved question in relevant logics.[16] As far as I can see, fuzzification does nothing to help the matter; but neither does it seem to make the question any harder.[17]

[15]We cannot obtain a proof theory for the finite case as in fn.10 since in FH the equivalence:

$$\{A_1, ..., A_n\} \vDash_{FH} B \text{ iff } \vDash_{FH} (A_1 \wedge ... \wedge A_n) \rightarrow B$$

fails from left to right. (The left hand side constrains behaviour only at normal worlds.) On the other hand, if one defines $A \oplus B$ as $\neg A \rightarrow B$, then $\oplus$ does not have the monotone properties required to refute compactness as in fn. 9, either.

[16]The best account on the market is, I think, Restall [7], which defines $*$ in terms of a primitive notion of incompatibility.

[17]Other paraconsistent logics can be fuzzified in the same way that H is fuzzified here.

2.5 Ternary Accessibility Relations

As relevant logics go, H is a relatively weak one. The standard way of making it stronger employs a ternary accessibility relation, R. This is added to interpretations, and the truth conditions for $\to$ *at non-normal worlds*, w, become:

$\nu_w(A \to B) = 1$ iff for all $y,\ z \in W$ such that $Rwyz$, if $\nu_y(A) = 1$ then $\nu_z(B) = 1$

A little thought suffices to show that this is equivalent to:

$$\nu_w(A \to B) = Glb\{\nu_y(A) \ominus \nu_z(B)\textbf{:}\ y, z \in W \text{ and } Rwyz\}$$

If we put no constraints at all on R, we have the relevant logic B. An axiom system for this is obtained from that for H by deleting R3-R5, and replacing them by the stronger:

A6 $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$

A7 $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$

R7 $A \to B, C \to D \vdash (B \to C) \to (A \to D)$

The semantics for B can be fuzzified in the obvious way, proceeding as before. Call the logic obtained in this way FB. It is more complex in this case to check that the new axioms/rules for B are valid/validity-preserving in FB; but it is true. I leave it as an exercise for the committed reader.[18] Hence the relationship between B and FB is the same as that between H and FH. Similar remarks also apply to its proof-theory.

Stronger logics in the relevant family are obtained by adding constraints on the ternary relation R. A novelty here is that the natural correspondence between constraints and axioms in standard relevant logics breaks down in some of the fuzzy cases. For example, in the standard case, adding the constraint that for all $w \in W$, $Rwww$, suffices to verify the axiom $(A \wedge (A \to$

Consider, for example, Da Costa's positive-plus logics. To fuzzify these, we start with a positive fuzzy logic, say Łukasiewicz'. We then add a non-truth-functional negation. The value of $\neg A$ is a number in $[0, 1]$ that is independent of the value of A (though one may put some constraints on the relationship between the two). The logics produced in this way are fuzzy and paraconsistent, but they are not relevant.

[18] Details can be found in Priest [5], ch. 11.

$B)) \rightarrow B$. This is no longer the case once we fuzzify. For example, consider the interpretation $\langle W, N, R, *, \nu\rangle$, where $W = \{g, x\}$; $N = \{g\}$; for every $w \in W$, $Rwww$; $\nu_x(p) = 0.8$ and $\nu_x(q) = 0.6$. Then $\nu_x(p \rightarrow q) = 0.8$. So $\nu_x(p \wedge (p \rightarrow q)) = 0.8$, and $\nu_g((p \wedge (p \rightarrow q)) \rightarrow q) \leq 0.8$. A full study of the connection between conditions on R and the corresponding axioms still needs to be undertaken for the fuzzy case.

As for the Routley $*$, there is a philosophical problem concerning the meaning of the ternary R. And as for $*$, fuzzification does nothing, as far as I can see, to help with this matter or to hinder it.

3 Strategy 2: Reinterpretation

3.1 De Morgan Lattices

Let us now turn to the second approach to the construction of a fuzzy relevant logic; and let us start by returning to standard fuzzy logic. A familiar criticism of this is that degrees of truth do not seem to be linearly ordered. 'Russell was old when he died' might have a higher degree of truth than 'Wittgenstein was old when he died'. But how does the degree of truth of 'Australia has a small population' relate to either of these?

The natural suggestion in response to this criticism is to trade in a linear order of truth values for a partial order. The values are no longer real numbers, of course. In fact, we may not care too much what the members of the order are. Let a semantic structure, then, be a partial order $\langle D, \leq\rangle$.

If sentences take values in this order, how do the connectives function? Conjunction and disjunction are easy. The order $\langle [0, 1], \leq\rangle$ of Łukasiewicz logic is a distributive lattice, and, in that lattice, Min and Max are the meet and join, respectively. Hence, the natural assumption is that the partial order is a distributive lattice, and that if $\nu(A)$ is the truth value of A then:

$$\nu(A \wedge B) = \nu(A) \wedge \nu(B)$$

$$\nu(A \vee B) = \nu(A) \vee \nu(B)$$

I write the lattice meet and join as $\wedge$ and $\vee$, respectively, context sufficing to disambiguate. I will employ the same convention for the algebraic operators corresponding to other connectives.

What of negation? In the semantics of Łukasiewicz logic, negation functions as in involution, that is, an order-inverting function of period two. We can generalise these aspects of its behaviour by supposing that our lattice comes with an operator, $\neg$, such that:

if $a \leq b$ then $\neg b \leq \neg a$

$\neg\neg a = a$

where $\nu(\neg a) = \neg\nu(a)$.

In summary, a natural generalisation of the semantic structure of a fuzzy logic for conjunction, disjunction and negation, once we drop the condition that it must be a linear order, is a distributive lattice with an involution. Such structures are known as De Morgan lattices. Moreover, if conjunction, disjunction and negation relate to the lattice in the way indicated, we have one of the well known semantics for a relevant logic. In particular, $A_1, ..., A_n \vDash B$ in First Degree Entailment iff $\nu(A_1 \wedge ... \wedge A_n) \leq \nu(B)$ for every De Morgan lattice, and every evaluation, ν, into the lattice.[19]

As a semantics for First Degree Entailment, the algebraic values would normally be thought of as propositions or Fregean senses, and $\leq$ would be thought of as some sort of containment relationship. But as we see, if we reconceptualise the interpretation of these notions, a fuzzy relevant logic falls straight out of the construction.[20]

3.2 De Morgan Groupoids

First Degree Entailment has no conditional connective. Can the preceding considerations be extended to cover such a connective? The most versatile algebraic semantics for relevant logic extends De Morgan lattices with new algebraic operators. A structure is now of the form $\langle \mathcal{D}, e, \circ, \rightarrow \rangle$, where $\mathcal{D}$ is a De Morgan lattice, $\circ$ and $\rightarrow$ are binary operators on the domain of the algebra, and e is a member of the domain. $\circ$ is standardly thought of as some sort of intensional conjunction, and e is thought of as the (value of the) conjunction of all truths. If the new components satisfy the following constraints:

$e \circ a = a$

[19]This characterisation does not account for inferences with an infinite number of premises; but this is no loss, since the logic is compact.

[20]For good measure, we might note that classical logic can be seen as a fuzzy logic in this way too, since it is sound and complete with respect to the class of Boolean algebras.

$a \circ b \leq c$ iff $a \leq b \to c$

if $a \leq b$ then $a \circ c \leq b \circ c$ and $c \circ a \leq c \circ b$

$a \circ (b \vee c) = (a \circ b) \vee (a \circ c)$ and $(b \vee c) \circ a = (b \circ a) \vee (c \circ a)$

then we have a structure called a De Morgan groupoid. And if we define $\Sigma \vDash B$ to mean that for all such groupoids, and all evaluations into its domain, ν, if $e \leq \nu(A)$ for all $A \in \Sigma$ then $e \leq \nu(B)$, we have a semantics sound and strongly complete with respect to the relevant logic B. In particular, A is a logical truth iff for all ν, $e \leq \nu(A)$. Stronger relevant logics in the family can be obtained by adding constraints on $\circ$.[21]

What sense can be made of this in fuzzy terms? For present purposes, $\circ$ can be thought of as an auxiliary notion. The crucial question therefore concerns e and $\to$. If we think of the members of the algebra as degrees of truth, then we may think of e as the lower bound of the things which are true enough to be acceptable. That is, $e \leq a$ iff something with value a is acceptable as true. What of $\to$? The algebraic postulates tell us that $e \leq a \to b$ iff $e \circ a \leq b$ iff $a \leq b$. Thus, a conditional is acceptable iff the truth value of the consequent is at least as great as that of the antecedent.

This is a plausible enough condition. One might have one's reservations about it, though. If, in a conditional, the truth value drops from antecedent to consequent, but only a very little, shouldn't it still be acceptable? (This is certainly how it works in standard fuzzy logics if $\varepsilon < 1$.) Perhaps not. On these semantics, we have to say that, strictly speaking, a conditional of this kind is not acceptable, though it might be as close to acceptable as

[21]It is worth noting that standard fuzzy logics (taking 1 as the only designated value) correspond to well know algebraic structures: MV algebras, G algebras, Π algebras, and BL algebras. The most basic of these are BL algebras, the collection of which characterises the logical truths common to all continuous t-norms. For details, see Hájek [2], chs. 3, 4. All these algebras are special cases of De Morgan groupoids. $\circ$ is, in fact, the connective whose truth conditions are given my the t-norm. In particular, a BL algebra is a groupoid in which $\circ$ is commutative and associative (as in some relevant logics), and which also satisfies the conditions:

$a \circ (a \to b) = a \wedge b$
$(a \to b) \vee (b \to a) = 1$

From a relevant point of view, the undesirablity of the second condition hardly needs to be pointed out. The first fails in all standard relevant logics as well. A logic called monoidal logic is given in Höhle [3]. This is characterised by the class of all residuated lattices. It is therefore a sublogic of B.

one might like. Indeed, given the possibility of sorites arguments, and the validity of *modus ponens* in the logic B, it has to be like this. At any rate, we see that, if we are prepared to live with the validity of *modus ponens* and the consequences of this, we can think of the relevant logic B, and its strengthenings, as already fuzzy logics.

3.3 Defeasible Conditionals

If one cannot live with these things, there is another way to proceed.[22] We add a new one-place function, δ, to the algebra, governed by the condition:

$\delta a \leq a$

Intuitively, δa is a value a little below a, and represents the value to which an antecedent would have to drop to make a conditional acceptable in the strong sense of the previous section. Employing this, we can define a different notion of conditionality, $a \triangleright b$, as $\delta a \to b$.

The easiest way to handle the new conditional proof-theoretically is to add a corresponding monadic functor, δ, to the language, augment the axioms with:

A8 $\delta A \to A$

R8 $A \leftrightarrow B \vdash \delta A \leftrightarrow \delta B$

and define $A \triangleright B$ as $\delta A \to B$. It is clear that the resulting axiom system is sound with respect to the semantics. A simple modification of the completeness proof for the algebraic semantics shows that it is complete also.[23] Let us call the logic obtained by extending B in this way DB (*defeasible* B). Stronger logics of the same kind can be obtained by modifying the algebras appropriate for stronger relevant logics in the same way.

The conditional $\triangleright$ in DB is a relevant one. For suppose that A and C share no propositional parameter. Then $\nvDash_B A \to C$. Consider a De Morgan groupoid counter-model. Extend this by defining δa as a. This is a model for DB, and in it, $\triangleright$ collapses into $\to$. Hence, $\nvDash_{DB} A \triangleright C$. But *modus ponens* for $\triangleright$ fails in DB and its extensions. To show this, we can construct a

[22]Hinted at in Sylvan and Hyde [10], p.13.

[23]Specifically, construct the Lindenbaum algebra in the usual way. R8 tells us that the function δ on the algebra may be defined in the standard fashion, and A8 gives δ its appropriate property.

counter-model as follows. Let $\mathcal{B}$ be the Boolean algebra of all subsets of ω. Let $\circ$ and $\rightarrow$ collapse into the corresponding extensional connectives (so that $e = \omega$). This is a De Morgan groupoid. (In fact, it is an algebra appropriate for every relevant logic.) Let δa be a with its least member removed (or if $a = \phi$, $\delta a = \phi$). Augmented by δ, the algebra is appropriate for DB (and the defeasible logics produced by using stronger relevant logics). To show that $p, p \rhd q \nvDash q$, set the values of p and q as ω and $\omega - \{0\}$, respectively.

The conditional $\rhd$ is also a defeasible one. That is, $p \rhd q \nvDash (p \wedge r) \rhd q$. For a counter-model, take the same algebra as before, except that:

$$\delta a = \begin{array}{ll} a - \{0\} & \text{if } 1 \in a \\ a - \{2\} & \text{otherwise} \end{array}$$

Take the values of p, q, and r, to be ω, $\omega - \{0\}$, and $\omega - \{1\}$, respectively.

In fact, DB has all the marks of a relevant conditional logic. In the world semantics for these, $A \rhd B$ is true at a world, w, iff $s(w, [A]) \subseteq [B]$, where $[C]$ is the set of worlds where C is true, and s is a function selecting subsets of W. If we impose the constraint on s that $s(w, [A]) \subseteq [A]$, we get the relevant conditional logic obtained by adding the following proof-theoretic rules to B.[24]

ID $A \rightarrow B \vdash A \rhd B$

REA $A \leftrightarrow B \vdash (A \rhd C) \rightarrow (B \rhd C)$

RPC $(A \wedge B) \rightarrow C \vdash ((D \rhd A) \wedge (D \rhd B)) \rightarrow (D \rhd C)$

It is easy to check that all these rules are sound in DB. I suspect that the rules are also complete (for the fragment without δ), but I have not been able to prove this yet.

It is not an implausible thought that the conditional involved in sorites arguments is a conditional of the kind given by DB: it certainly does not seem to be an entailment. At any rate, the construction that we have just been considering gives us another fuzzy relevant logic.[25]

[24] See Sylvan [9].

[25] Another algebraic way of proceeding is by modifying the notion of a De Morgan groupoid. Everything is as in 3.2, except that the condition $e \circ a = a$ is weakened to $e \circ a \leq a$. In effect, $e \circ a$ now plays the role of δa. The logic obtained in this way is clearly a relevant logic, and, as may be checked, *modus ponens* for $\rightarrow$ fails. But $\rightarrow$ is not the same as $\rhd$ in DB, since $\rightarrow$ is not defeasible. For suppose that $e \leq a \rightarrow b$. Then $e \circ a \leq b$. But $a \wedge c \leq a$. So $e \circ (a \wedge c) \leq e \circ a$. It follows that $e \circ (a \wedge c) \leq b$, i.e., $e \leq (a \wedge c) \rightarrow b$.

4 Conclusion

In this paper, we have examined two semantic strategies for constructing fuzzy relevant logics. The first fuzzifies standard world-semantics for relevant logics, changing the discrete truth values to continuum-valued ones. This construction gives sublogics of the corresponding standard relevant logics. In particular, *modus ponens* is no longer valid. This is as one might expect in a fuzzy logic where 1 is not taken to be the only designated value. In the second strategy, we simply reinterpret the algebraic semantics for relevant logic, thinking of the algebraic values as degrees of truth. The upshot of this is that standard relevant logics can already be thought of as fuzzy logics. In particular, then, in these semantics *modus ponens* holds. These semantics can be extended by a "decrease in value" operator, δ, which can be used to define a defeasible relevant conditional, for which, again, *modus ponens* fails. Which, if any, of these logics is philosophically the best for their intended application is another matter. But at least we now have some fuzzy relevant logics to philosophise about.[26]

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