distributed with location μ_i on a subjective continuum (cf. also **Normal distribution**). A special case gives that the probability that T_i is preferred to T_j is equal to

$$\mathsf{P}(T_i > T_j) = \mathsf{P}(X_i > X_j) = \frac{1}{\sqrt{2\pi}} \int_{-(\mu_i - \mu_j)}^{\infty} e^{-y^2/2}\, dy.$$

If the normal density function is replaced by the logistic density function, the model is equal to the Bradley-Terry model with $\mu_i = \ln \pi_i$. H. Stern has considered, [10], models for paired comparison experiments based on comparison of gamma random variables. Different values of the shape parameter yield different models, including the Bradley–Terry model and the Thurstone model. Likelihood methods can be used to estimate the parameters of the models. The likelihood equations must be solved with iterative methods.

It is also possible to fit response surfaces in paired comparison experiments (see, e.g., [9], [2]). Mostly it is assumed that the parameters π_i, $i = 1, \dots, t$, are functions of continuous variables $x_1, \dots, x_s$ such that the formulated model is linear in the unknown parameters β_j. If such a model is formulated, then it is possible to discuss the question of optimal design in paired comparison experiments. Many criteria for optimal design depend on the variance-covariance matrix of the estimators for the unknown parameters β_j. However, the asymptotic variance-covariance matrix itself depends on the unknown parameters (see, e.g., [9], [2]). A. Springall has defined, [9], so-called *analogue designs*. These are designs in which the elements of the paired comparison variance-covariance matrix are proportional to the elements of the classical response surface variance-covariance matrix with the same design points.

In order to find designs, E.E.M. van Berkum has assumed, [2], that the parameters β_j are all equal. In that case the variance-covariance matrix is proportional to the variance-covariance matrix for the estimators in an ordinary linear model and general optimal design theory can be applied (D-optimality, G-optimality, equivalence theorem). He also gives optimal designs for various factorial models.

There is much literature on paired comparison experiments and related topics such as generalized linear models, log-linear models, weighted least squares and non-parametric methods. A bibliography up to 1976 is given in [7]. The state of the art as of 1976 is given in [3], and as of 1992 in [5].

References

[1] BEAVER, R.J., AND GOKHALE, D.V.: 'A model to incorporate within-pair order effects in paired comparisons', *Commun. in Statist.* **4** (1975), 923–929.

[2] BERKUM, E.E.M. VAN: *Optimal paired comparison designs for factorial experiments*, Vol. 31 of *CWI Tract*, CWI, Amsterdam, 1987.

[3] BRADLEY, R.A.: 'Science, statistics and paired comparisons', *Biometrics* **32** (1976), 213–232.

[4] BRADLEY, R.A., AND TERRY, M.E.: 'The rank analysis of incomplete block designs. I. The method of paired comparisons', *Biometrika* **39** (1952), 324–345.

[5] DAVID, H.A.: 'Ranking and selection from paired-comparison data. With discussion': *The Frontiers of Modern Statistical Inference Procedures II (Sydney, 1987)*, Vol. 28 of *Math. Management Sci.*, Amer. Sci. Press, 1992, pp. 3–24.

[6] DAVIDSON, R.R.: 'On extending the Bradley–Terry model to accommodate ties in paired comparison experiments', *J. Amer. Statist. Assoc.* **65** (1970), 317–328.

[7] DAVIDSON, R.R., AND FARQUHAR, P.H.: 'A bibliography on the method of paired comparisons', *Biometrika* **32** (1976), 241–252.

[8] RAO, P.V., AND KUPPER, L.L.: 'Ties in paired-comparison experiments: A generalization of the Bradley–Terry model', *J. Amer. Statist. Assoc.* **62** (1967), 194–204.

[9] SPRINGALL, A.: 'Response surface fitting using a generalization of the Bradley-Terry paired comparison model', *Appl. Statist.* **22** (1973), 59–68.

[10] STERN, H.: 'A continuum of paired comparison models', *Biometrika* **77** (1990), 265–273.

[11] THURSTONE, L.L.: 'Psychophysical analysis', *Amer. J. Psychol.* **38** (1927), 368–389.

E.E.M. van Berkum

MSC 1991: 62J15, 62K05

PARACONSISTENT LOGIC – A relation of logical consequence, $\vdash$, on a set of sentences, S, is *explosive* if and only if for all $\alpha, \beta \in S$,

$$\alpha, \neg\alpha \vdash \beta,$$

where '$\neg$' is negation. A relation, and the logic that possesses it, is *paraconsistent* if and only if it is not explosive. Whether or not a correct consequence relation is explosive has been a contentious issue historically, but the standard formal logics of the 20th century, such as classical logic (cf. **Logical calculus**) and **intuitionistic logic** are explosive. Formal paraconsistent logics were developed by a number of different people, often working in isolation from each other, starting around the 1960s.

There are many different paraconsistent logics, with their own proof theories and model theories. Their distinctive features occur at the propositional level, though they all have full first- (and second-) order versions. In most of them validity can be defined in terms of preservation of truth in an interpretation.

In one approach, due to S. Jaśkowski, an interpretation is a Kripke model (cf. **Kripke models**) for some **modal logic**, and a sentence is true in it if it holds at some world of the interpretation. A major feature of this approach is that the inference of *adjunction* ($\alpha, \beta \vdash \alpha \wedge \beta$) fails. In another, an interpretation ν, is a mapping from S to $\{1, 0\}$, satisfying the usual classical conditions for $\wedge$, $\vee$, and $\to$. $\nu(\neg\alpha)$ is independent of $\nu(\alpha)$. The addition of further constraints on ν, such as:

$\nu(\alpha) = 0 \Rightarrow \nu(\neg\alpha) = 1$, give logics in N. da Costa's C *family*. A feature of this approach is that it preserves all of positive logic. In a third approach, an interpretation ν is a mapping from S to the closed sets of a **topological space** $\mathcal{T}$ satisfying the conditions $\nu(\alpha\wedge\beta) = \nu(\alpha)\cap\nu(\beta)$, $\nu(\alpha \vee \beta) = \nu(\alpha) \cup \nu(\beta)$, $\nu(\neg\alpha) = \overline{\nu(\alpha)}^c$ (where c is the closure operator of $\mathcal{T}$). α is true under ν if and only if $\nu(\alpha)$ is the whole space. This gives a logic dual to **intuitionistic logic**.

In a fourth approach, an interpretation is a relation $\rho \subseteq S \times \{1,0\}$, satisfying the natural conditions

$$\neg\alpha\rho 1 \Leftrightarrow \alpha\rho 0,$$
$$\neg\alpha\rho 0 \Leftrightarrow \alpha\rho 1;$$
$$\alpha \wedge \beta\rho 1 \Leftrightarrow \alpha\rho 1 \text{ and } \beta\rho 1,$$
$$\alpha \wedge \beta\rho 0 \Leftrightarrow \alpha\rho 0 \text{ or } \beta\rho 0;$$

and dually for $\vee$. α is *true* under ρ if and only if $\alpha\rho 1$. This gives the logic of first degree entailment (FDE) of A. Anderson and N. Belnap. If one restricts interpretations to those satisfying the condition $\forall\alpha\exists x \alpha\rho x$, one gets G. Priest's LP. A feature of this logic is that its logical truths coincide with those of classical logic. Thus, the law of non-contradiction holds: $\vdash \neg(\alpha \wedge \neg\alpha)$. A *De Morgan lattice* is a **distributive lattice** with an additional operator $\neg$ satisfying: $\neg\neg a = a$ and $a \leq b \Rightarrow \neg b \leq \neg a$. An FDE-interpretation can be thought of as a homomorphism into the De Morgan lattice with values $\{\{1\},\{1,0\},\emptyset,\{0\}\}$. More generally, $\alpha \vdash \beta$ in FDE if and only if for every homomorphism h into a De Morgan lattice, $h(\alpha) \leq h(\beta)$. Augmenting such lattices with an operator $\rightarrow$ satisfying certain conditions, and defining validity in the same way, gives a family of relevant logics.

A paraconsistent logic localizes contradictions, and so is appropriate for reasoning from information that may be inconsistent, e.g., information stored in a computer database. It also permits the existence of theories (sets of sentences closed under deducibility) that are inconsistent but not *trivial* (i.e., containing everything) and of their models, inconsistent structures.

One important example of an *inconsistent theory* is set theory based on the general *comprehension schema* ($\exists x\forall y(y \in x \leftrightarrow \alpha)$, where α is any formula not containing x), together with *extensionality* ($\forall x(x \in y \leftrightarrow x \in z) \vdash y = z$). Another is a theory of truth (or of other semantic notions), based on the *T-schema* ($T\langle\alpha\rangle \leftrightarrow \alpha$, where α is any closed formula, and $\langle\cdot\rangle$ indicates a name-forming device), together with some mechanism for self-reference, such as arithmetization. Such theories are inconsistent due to the paradoxes of self-reference (cf. **Antinomy**).

Not all paraconsistent logics are suitable as the underlying logics of these theories. In particular, if the underlying logic contains *contraction* ($\alpha \rightarrow (\alpha \rightarrow \beta) \vdash \alpha \rightarrow \beta$) and *modus ponens* ($\alpha, \alpha \rightarrow \beta \vdash \beta$), these theories are trivial. However, the theories are non-trivial if $\rightarrow$ is interpreted as the material conditional and the logic LP is used, or if it is interpreted as the conditional of some relevant logics. In the truth theories, the inconsistencies do not spread into the arithmetical machinery.

Given a **topos**, logical operators can be defined as functors within it, and a notion of internal validity can be defined, giving intuitionistic logic. If these operators, and in particular, negation, are defined in the dual way, the internal logic of the topos is the *dual intuitionistic logic*. Topoi can therefore be seen as inconsistent structures.

For another example of inconsistent structures, let A be the set of sentences true in the standard model of arithmetic. If B is a set of sentences in the same language properly containing A, then B is inconsistent, and so has no classical models; but B has models, including finite models, in the paraconsistent logic LP. Inconsistent (sets of) equations may have solutions in such models. The LP-models of A include the classical non-standard models of arithmetic (cf. **Peano axioms**) as a special case, and, like them, have a notable common structure.

In inconsistent theories of arithmetic, the incompleteness theorems of K. Gödel (cf. **Gödel incompleteness theorem**) fail: such a theory may be axiomatizable and contain its own 'undecidable' sentence (and its negation).

Inconsistent theories may be interesting or useful even if they are not true. The view that some inconsistent theories are true is called *dialetheism* (or *dialethism*).

For a general overview of the area, see [2]. [3] is a collection of articles, with much background material. On inconsistent mathematical structures, see [1].

References

[1] MORTENSEN, C.: *Inconsistent mathematics*, Kluwer Acad. Publ., 1995.

[2] PRIEST, G.: 'Paraconsistent logic', in D. GABBAY AND F. GUENTHNER (eds.): *Handbook of Philosophical Logic*, Vol. VII, Kluwer Acad. Publ., forthcoming.

[3] PRIEST, G., ROUTLEY, R., AND NORMAN, G. (eds.): *Paraconsistent logic: essays on the inconsistent*, Philosophia Verlag, 1989.

G. Priest

MSC 1991: 03B53

PARTIALLY SPECIFIED MATRICES, COMPLETION OF – A *partially specified* $(p \times q)$*-matrix* is a $(p \times q)$-array of complex numbers (or, more generally, of elements over an arbitrary **field**) in which certain