Fusion and Confusion

Graham Priest

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1 Introduction

Curry's paradox is well known.[1] It comes in both set theoretic and semantic versions. Here we will concentrate on the semantic versions. Historically, these have deployed the notion of truth. Those who wish to endorse an unrestricted T-schema have mainly endorsed a logic which rejects the principle of Absorption, $A \to (A \to B) \models A \to B$. High profile logics of this kind are certain relevant logics; these have semantics which show how and why this principle is not valid. Of more recent times, paradoxes which are clearly in the same family have been appearing; but these concern the notion of validity itself. The standard semantics of relevant logics seem powerless to address these. But they can. This note shows how. The upshot can be seen as a return to the roots of relevant logic, in a sense to become clear.

2 Background

2.1 The Usual Curry Paradoxes

Let us start with a couple of standard forms of the paradox. It will be useful to formulate them in terms of natural deduction arguments, operating on sequents of the form $X \rhd A$, where X is a set of premises, and A is the conclusion. When enumerating the members of X I will omit set braces. One may read '$X \rhd A$' as 'X implies A'.

The logic has axioms of the form:

$$A \rhd A$$

the rules of *modus ponens* (MP):

[1]See, e.g., Priest (2006), ch. 6.

$$\frac{X \rhd A \to B \qquad Y \rhd A}{X \cup Y \rhd B}$$

the conjunction rules (CR):

$$\frac{X \rhd A \wedge B}{X \rhd A\,(B)} \qquad\qquad \frac{X \rhd A \quad Y \rhd B}{X \cup Y \rhd A \wedge B}$$

and Cut:

$$\frac{X \rhd A \qquad Y, A \rhd B}{X \cup Y \rhd B}$$

The theory of truth built on this has as axioms all instances of the T-schema:

$$\rhd\, T\langle A\rangle \leftrightarrow A$$

where $A \leftrightarrow B$ is defined as $(A \to B) \wedge (B \to A)$. We assume, also, that we have some form of self-reference which allows us to construct a sentence, C, of the form $T\langle C\rangle \to \bot$.[2]

The first version of the argument[3] assumes the validity of the Absorption schema (Abs):[4]

$$A \to (A \to B) \rhd (A \to B)$$

The argument then goes as follows.

1	$T\langle C\rangle \leftrightarrow C$	$\rhd$	$T\langle C\rangle \leftrightarrow C$	
2	$T\langle C\rangle \leftrightarrow C$	$\rhd$	$T\langle C\rangle \to C$	CR
3	$T\langle C\rangle \leftrightarrow C$	$\rhd$	$T\langle C\rangle \to (T\langle C\rangle \to \bot)$	
4	$T\langle C\rangle \leftrightarrow C$	$\rhd$	$T\langle C\rangle \to \bot$	Abs and Cut
5	$T\langle C\rangle \leftrightarrow C$	$\rhd$	C	
6	$T\langle C\rangle \leftrightarrow C$	$\rhd$	$T\langle C\rangle \leftrightarrow C$	
7	$T\langle C\rangle \leftrightarrow C$	$\rhd$	$C \to T\langle C\rangle$	CR
8	$T\langle C\rangle \leftrightarrow C$	$\rhd$	$T\langle C\rangle$	5, 7 and MP
9	$T\langle C\rangle \leftrightarrow C$	$\rhd$	$\bot$	8, 4 and MP
10		$\rhd$	$\bot$	T-schema and Cut

[2] $\bot$ can be taken to be governed by the axiom $\bot \rhd A$, or simply replaced with any sentence whatsoever.

[3] Curry (1942).

[4] This is often called Contraction, but I will reserve that name for the structural rule, to be met later.

The second form of Curry's paradox[5] uses the principle of *pseudo modus ponens* (*PMP*):

$$\rhd (A \wedge (A \rightarrow B)) \rightarrow B$$

together with one further fact about conjunction, idempotence (Idem):

$$\rhd A \leftrightarrow (A \wedge A)$$

and a rule of substitution (Subst):

$$\frac{X \rhd \varphi(A)}{X, A \leftrightarrow B \rhd \varphi(B)}$$

where $\varphi(A)$ is any context in which A occurs.

We then have:

1		$\rhd$	$(C \wedge (T\langle C\rangle \rightarrow \bot)) \rightarrow \bot$	PMP
2		$\rhd$	$(C \wedge C) \rightarrow \bot$	
3	$C \leftrightarrow (C \wedge C)$	$\rhd$	$C \rightarrow \bot$	Subst
4		$\rhd$	$C \rightarrow \bot$	Idem and Cut
5	$T\langle C\rangle \leftrightarrow C$	$\rhd$	$T\langle C\rangle \rightarrow \bot$	Subst
6	$T\langle C\rangle \leftrightarrow C$	$\rhd$	C	
7	$T\langle C\rangle \leftrightarrow C$	$\rhd$	$T\langle C\rangle$	Subst
8	$T\langle C\rangle \leftrightarrow C$	$\rhd$	$\bot$	$5, 7, MP$
9		$\rhd$	$\bot$	T-schema and Cut

2.2 Semantics

A standard solution to Curry's paradox, by those who want to endorse an unrestricted T-schema,[6] is to reject *PMP* and *Abs*. Perhaps the simplest and most robust justification for this is to use the semantical technology of impossible worlds. In Routley/Meyer semantics, this works as follows.[7]

We take a language that contains the connectives $\rightarrow$ and $\wedge$. As we will see, it will be helpful to take the language to contain an intensional conjunction, $\circ$ ("fusion"), and a logical constant, t—thought of as the conjunction of all

[5] Meyer, Routley, and Dunn (1979).

[6] Such as Priest (2006), Field (2008), and Beall (2009).

[7] There is some flexibility about how, exactly, to set up the semantics, and what, exactly, constitutes the logic B. The following is the approach of Routley, *et al.* (1982), chs. 4, 5, except that they allow for a *set* of base worlds, P, and define $x \leq y$ as: for some $w \in P$, $Rwxy$. This does not affect the set of logical truths. A slightly different approach (the "simplified semantics") is given in Priest (2008), ch. 10.

logical truths—as well. There may other connectives (and quantifiers) in the language, but they are not relevant to the story, so I will ignore them in what follows.

An interpretation is a structure $\langle W, @, R, \nu\rangle$, where W is a set (of worlds), @ is the "base world"; R is a ternary relation on W; and ν maps each world, w, and parameter, p, to a value $\nu_w(p) \in \{1, 0\}$. We define $x \leq y$ as: $R@xy$. Then the interpretation must satisfy the following conditions, for all parameters, p, and worlds, a, b, x, y:

- $a \leq a$
- if $a \leq b$ then if $\nu_a(p) = 1$ then $\nu_b(p) = 1$
- if $a \leq b$ then if $Rbxy$ then $Raxy$
- if $a \leq b$ then if $Rxya$ then $Rxyb$

The last three clauses, together with the truth conditions for the connectives, ensure that if $a \leq b$ then anything true at a is true at b. Indeed, one can think of this as the meaning of '$\leq$'.

Given an interpretation, ν is extended to an evaluation of all formulas at worlds by the recursive clauses:

- $\nu_w(t) = 1$ iff $@ \leq w$
- $\nu_w(A \wedge B) = 1$ iff $\nu_w(A) = \nu_w(B) = 1$
- $\nu_w(A \circ B) = 1$ iff for some x, y in W such that $Rxyw$, $\nu_x(A) = 1$ and $\nu_y(B) = 1$
- $\nu_w(A \to B) = 1$ iff for all x, y in W, such that $Rwxy$, if $\nu_x(A) = 1$ then $\nu_y(B) = 1$

An inference with premises Σ and conclusion A is valid, $\Sigma \models A$, iff for every interpretation, if $\nu_@(B) = 1$, for all $B \in \Sigma$, then $\nu_@(A) = 1$. $\models A$ iff $\emptyset \models A$. It is also easy to check that $\emptyset \models A$ iff $t \models A$. These semantics give the ($\to$, $\wedge$, $\circ$ fragment) of the basic relevant logic **B**. Further constraints on R give stronger logics.

The semantics validate the sequent rules so far used, in the sense that if '$\rhd$' is replaced by '$\models$', and the premises are true (at @), so is the conclusion. However, they verify neither *Abs* nor *PMP*. It is not hard to show that both $A \to (A \to B) \not\models A \to B$ and $\not\models (A \wedge (A \to B)) \to B$. The trick in both cases is the deployment of worlds which are not closed under *modus ponens*. There are (impossible) worlds where A and $A \to B$ are true and B is not.

The detailed construction of counter-models is left as an exercise. It can also be shown that, given this machinery, *Abs* is equivalent to PMP, exposing a connection between the two versions of the curry paradox.

It should be emphasized that none of this threatens the validity of *modus ponens*. It is easy to check that $A, A \to B \models B$. This follows from the fact that $R@@@$. The validity of *Abs* and PMP, note, requires the condition that for all $w \in W$, $Rwww$.

3 The Problem: Validity Curry

So far so good. However, of late, paradoxes clearly in the same family as Curry's paradox, deploying the notion of validity, have been turning up. Thus, one thing that one should clearly expect from validity is that a valid inference be truth-preserving. Thus, if we express the validity of a single-premise inference from A to B as $V(\langle A\rangle \langle B\rangle)$, then we would expect to have: $V(\langle C\rangle \langle D\rangle) \rhd T\langle C\rangle \to T\langle D\rangle$.

But as I have noted, *modus ponens* is valid: $A, A \to B \models B$;[8] so we have $\rhd V(\langle A \wedge (A \to B)\rangle \langle B\rangle)$, and so by Cut, $\rhd T\langle A \wedge (A \to B)\rangle \to T\langle B\rangle$. The T-schema and Subst then give *PMP*, and we are back with Curry's paradox.[9]

In fact, truth is playing no essential role in this argument. We would equally expect a notion of validity to give us $V(\langle C\rangle \langle D\rangle) \rhd C \to D$, which delivers *PMP* even faster. And given this, we can formulate a version of Curry's paradox directly. Validity would seem to be governed by the rules, V1 and V2, respectively:

$$\frac{}{V(\langle C\rangle \langle D\rangle) \rhd C \to D} \qquad\qquad \frac{C \rhd D}{\rhd V(\langle C\rangle \langle D\rangle)}$$

Given the usual self-reference, we can construct a sentence, C, of the form $V(\langle C\rangle \langle \perp\rangle)$.

[8] In fact, one can establish this using just the sequent rules cited. Details are left as an exercise.

[9] See, e.g., Field (2008).

We then have:[10]

1	$V(\langle C\rangle\,\langle\bot\rangle)$	$\rhd$	$V(\langle C\rangle\,\langle\bot\rangle)$	
2	C	$\rhd$	$V(\langle C\rangle\,\langle\bot\rangle)$	
3	C	$\rhd$	$C\to\bot$	$V1$, Cut
4	C	$\rhd$	C	
5	C	$\rhd$	$\bot$	$3, 4, MP$
6		$\rhd$	$V(\langle C\rangle\,\langle\bot\rangle)$	$5, V2$
7		$\rhd$	C	
8		$\rhd$	$C\to\bot$	$6, V1,$ Cut
9		$\rhd$	$\bot$	$7, 8, MP$

Call this argument (*) for future reference.

The rejection of *Abs* and *PMP* does nothing to help defuse these arguments. As we noted, the standard semantics validates *modus ponens*; valid inferences preserves truth at @ (actual truth), $V1$; and $V2$ appears little more than definitional.

The astute reader will have observed, however, that in the deduction (*) we have used the assumption C twice in the deduction of $\bot$ at line 5. It looks as though some kind of contraction is involved in this. But the contraction has to do with the use of premises, not antecedents of conditionals. The standard semantics conceptualises validity as a relation between premises, thought of as sets of sentences, and a conclusion which is a sentence. And contraction for sets is trivial: $\{A, A\} = \{A\}$. How, then, can one make sense of failures of premise contraction semantically? The rest of the paper shows how.

4 Background to the Solution: Substructural Proof-Theory

We take our cue from a substructural proof theory.[11] Sequents are still of the form $X \rhd A$, but the Xs are different. Before, we had only one way of combining formulas into collections of premises: an extensional form, which combines premises into sets. Now we will have two, the extensional one, which we will write as $\oplus$, and an intensional one, which we will write as $\otimes$. I will come back to the meaning of $\otimes$ in due course. For the nonce, one can think of it as some sort of intentional conjoining. Collections of formulas

[10] Beall and Murzi (201+). See also Whittle (2004).

[11] Details can be found in Slaney (1990), Read (1988), chs. 4, 5, and Restall (2000), chs. 2, 11.

obtained by wielding these two methods of combination are called *bunches*. Formally, bunches are the smallest class generated by the following rules:

- **1** is a bunch.
- Any formula is a bunch.
- If X and Y are bunches then $(X \oplus Y)$ and $(X \otimes Y)$ are bunches.

I will omit outermost parentheses. **1** is a bunch that corresponds to the constant t, in a sense to be made precise in a moment.

Axioms of the deduction system are sequents of the form:

$$A \rhd A$$

The rules governing the connectives are as follows:

$$\frac{X \rhd A \quad X \rhd B}{X \rhd A \wedge B} \qquad \frac{X \rhd A \wedge B}{X \rhd A\,(B)}$$

$$\frac{X \otimes A \rhd B}{X \rhd A \to B} \qquad \frac{Y \rhd A \to B \quad X \rhd A}{Y \otimes X \rhd B}$$

$$\frac{X \rhd A \quad Y \rhd B}{X \otimes Y \rhd A \circ B} \qquad \frac{X \rhd A \circ B \quad Y \rhd C}{Y_{A \otimes B}(X) \rhd C}$$

$$\frac{}{\mathbf{1} \rhd t} \qquad \frac{X \rhd t \quad Y \rhd A}{Y_{\mathbf{1}}(X) \rhd A}$$

where $X_Y(Z)$ means the bunch X with any or all occurrences of the bunch Y replaced by the bunch Z.

As well as the rules for the connectives, we have structural rules concerning the bunches. I will write such a rule in the form $X \mapsto Y$. This is to be understood as:

$$\frac{Z \rhd A}{Z_X(Y) \rhd A}$$

The structural rules may be formulated as follows:

$$\begin{array}{cccc}
X \oplus (Y \oplus Z) & \mapsto & (X \oplus Y) \oplus Z & \text{Associativity} \\
X \oplus Y & \mapsto & Y \oplus X & \text{Commutativity} \\
X \oplus X & \mapsto & X & \text{Contraction} \\
X & \mapsto & X \oplus Y & \text{Weakening} \\
X & \mapsto & \mathbf{1} \otimes X & \text{Push} \\
\mathbf{1} \otimes X & \mapsto & X & \text{Pop}
\end{array}$$

The first three of these effectively turn $\oplus$ into a set-forming (as opposed to a sequence-forming, multiset-forming, or list-forming) operator. The next is a version of monotonicity. The last two give us the distinctive properties of **1**. We may also have a rule of Cut, although this is eliminable:[12]

$$\frac{X \rhd A \quad Y \rhd B}{Y_A(X) \rhd B}$$

Examples of proofs with these various rules can be found in the references cited.

Note that there is no bunch which corresponds to the empty set of premises. The is no sequent of the form $\rhd A$. (Nothing follows from nothing!) What plays the role of the empty set of premises, is the bunch **1**. We may therefore express the thought that A follows from the empty set of premises as $\mathbf{1} \rhd A$.

We may now define a map from bunches to formulas, #, by recursion:

- $\mathbf{1}^\#$ is t
- $A^\#$ is A
- $(X \oplus Y)^\#$ is $X^\# \wedge Y^\#$
- $(X \otimes Y)^\#$ is $X^\# \circ Y^\#$

The definition allows us to establish the connection between $\otimes$ and $\circ$, on the one hand, and $\oplus$ and $\wedge$, on the other, namely: $X \rhd A$ is inter-derivable

[12]See the cited reference by Restall.

with $t \rhd X^{\#} \to A$. The proof is by induction on the structure of X in both directions, and I leave the details as an exercise.[13]

We can also spell out the connection between the semantics and the deduction system. Given an interpretation, I, say that A *suffices for* B *in* I iff for every world, w, if A is true at w, so is B. It is not difficult to show that for any I, A suffices for B iff $A \to B$ is true at @ (in I). Say that A *suffices for* B (period) iff A suffices for B in every interpretation, i.e., $A \to B$ is a logical truth.

The soundness and completeness theorem for the logic **B** tells us that $X \rhd A$ is provable iff $X^{\#}$ suffices A.[14] Hence:

$$\begin{array}{rll} \models X^{\#} \to A & \text{iff} & X^{\#} \text{ suffices for } A \\ & \text{iff} & X \rhd A \text{ is provable} \end{array}$$

I note that the result carries over to logics stronger than **B**, when further structural rules are added. Thus, the addition of the analogues of any of the first three structural rules with $\otimes$ instead of $\oplus$ delivers stronger relevant logics. (All three together give the logic **R**.) Adding the analogue of the fourth rule delivers a proof of $t \rhd A \to (B \to A)$, and so a logic which is not a relevant logic.

Given the importance of *Abs* and *PMP* in the present context, let us note their (non-)proofs in the logic **B**. (The may be provable in stronger logics.) For *Abs*:

$$\begin{array}{lllll} 1 & A \to (A \to B) & \rhd & A \to (A \to B) & \\ 2 & A & \rhd & A & \\ 3 & (A \to (A \to B)) \otimes A & \rhd & A \to B & MP \\ 4 & A & \rhd & A & \\ 5 & ((A \to (A \to B)) \otimes A) \otimes A & \rhd & B & MP \\ 6 & (A \to (A \to B)) \otimes A & \rhd & B & ? \\ 7 & A \to (A \to B) & \rhd & A \to B & \end{array}$$

The argument fails without a structural rule of the form $(X \otimes Y) \otimes Y \mapsto X \otimes Y$. This is needed at line 6.

For *PMP*, let X be $A \wedge (A \to B)$. Then:

[13] In standard presentation of linear logic, there are the two notions of conjunction in the language, but only one form of premise combination, the intentional one. This gives rise to certain features, such as the failure of distribution for the extensional connectives. Given the connection between the conjunctions and premise combination, it would seem to me to be much more natural to have both forms of premise combination (and so distribution), in the way that I have set things up.

[14] I omit the details of proof here. They can be extracted from the references cited.

$$\begin{array}{lll} 1 & X & \rhd & A \\ 2 & X & \rhd & A \to B \\ 3 & X \otimes X & \rhd & B \\ 4 & X & \rhd & X \to B \\ 5 & \mathbf{1} \otimes X & \rhd & X \to B \\ 6 & \mathbf{1} & \rhd & X \to (X \to B) \end{array}$$

All we can show is an uncontracted version of the principle. To show *PMP* itself, we would need the structural rule $X \otimes X \mapsto X$, to be applied after line 3.

5 The Solution

5.1 Validity Curries Revisited

Armed with this material, we can now address the validity curries.

Given the proof-theory, the natural definition of validity is that an inference with premise, X, and conclusion, A, is valid if $X \rhd A$ is true. Now:

$$\begin{array}{lll} X \rhd A \text{ is provable} & \text{iff} & X^{\#} \text{ suffices for } A \\ & \text{iff} & \models X^{\#} \to A \\ & \text{iff} & t \rhd (X^{\#} \to A) \text{ is true} \\ & \text{iff} & X \rhd A \text{ is true} \end{array}$$

(For lines 2 and 3, if $C \to D$ is a logical truth, then it follows from the conjunction of all logical truths; and if it follows from the conjunction of all logical truths, it is a logical truth.) Hence, all these are equivalent ways of saying that an inference is valid.

The problem of truth preservation is now solved because *modus ponens* in the appropriate form is not valid. It is not the case that $\models A \wedge (A \to B) \to B$.

For the argument (*), V1 is fine. It records the fact that a valid inference preserves truth, which is correct: if $A \to B$ is a logical truth, then A suffices for B. V2 now becomes:

$$\frac{C \rhd D}{\mathbf{1} \rhd V(\langle C \rangle \, \langle D \rangle)}$$

which is true by definition.[15]

[15] I note that as long as there are no quantifiers in the language, one can interpret the system with the V rules in the one without, simply by translating $V(\langle C \rangle \, \langle D \rangle$ as $C \to D$.

And the argument (*) now looks like this:

1	$V(\langle C\rangle\,\langle\bot\rangle)$	$\rhd$	$V(\langle C\rangle\,\langle\bot\rangle)$	
2	C	$\rhd$	$V(\langle C\rangle\,\langle\bot\rangle)$	
3	C	$\rhd$	$C\to\bot$	$V1$, Cut
4	C	$\rhd$	C	
5	$C\otimes C$	$\rhd$	$\bot$	$3, 4, MP$
6	C	$\rhd$	$\bot$	?
7	$\mathbf{1}$	$\rhd$	$V(\langle C\rangle\,\langle\bot\rangle)$	$5, V2$
8	$\mathbf{1}$	$\rhd$	C	
9	$\mathbf{1}$	$\rhd$	$C\to\bot$	$6, V1$, Cut
10	$\mathbf{1}\otimes\mathbf{1}$	$\rhd$	$\bot$	$7, 8, MP$
11	$\mathbf{1}$	$\rhd$	$\bot$	Pop

Call this argument (**). It involves an illicit contraction at line 6.

5.2 Reasoning from Information

So does this mean that one cannot use *modus ponens*? No. *Modus ponens* is one of the rules of inference. Given a proof of A and a proof of $A\to B$, we have a proof of B. But B will depend on whatever A depends on "fuse" whatever $A\to B$ depends on.

This does have a implications for how we are to understand what it is to be provable from some information, such as that provided by the axioms of an axiom system, however. Normally, this is specified just as a list of formulas. How is it to be understood that we may combine these: extensionally, intensionally, in what order? The answer is that the specification should be understood as entitling us to *any* bunch made up of the formulas on the list (and **1**). Specifically, if Σ is a set of sentences, and $\mathfrak{B}(\Sigma)$ is the set of bunches formed from members of Σ, we are allowed to help ourselves to any member of $\mathfrak{B}(\Sigma)$. That is, A follows from Σ iff for some $X\in\mathfrak{B}(\Sigma)$, $X\rhd A$ is provable. Given this, when inferring from the axioms, we can use *modus ponens* (and adjunction) with impunity.[16]

It might be thought that we now have a "revenge problem". Suppose that we define $Val(\langle A\rangle\,,\langle B\rangle)$ as: for some $X\in\mathfrak{B}(A)$, $V(\left\langle X^{\#}\right\rangle,\langle B\rangle)$; and then we run argument (**) with Val instead of V. The move from lines

[16] In his (1990), Slaney moots a solution to the sorites paradox. He notes, in effect, that given the premises of a sorites inference, $\Sigma=\{A_0,\ A_1\to A_2,\ \dots\ ,\ A_{n-1}\to A_n\}$, the deductive machinery allows us to establish only that $(\prod_{i=n}^{1}(A_i\to A_{i-1})\otimes A_0)\to A_n$ (associating to the right). Given that we cannot accept A_n, we are not entitled to an arbitrary bunch in $\mathfrak{B}(\Sigma)$. The premises cannot be taken together, as it were, in the appropriate fashion.

5 to 7 of the argument is now fine. The problem is with V1. Clearly, we should not have $Val(\langle C\rangle, \langle\bot\rangle) \rhd C \to \bot$. Effectively, $Val(\langle C\rangle, \langle\bot\rangle)$ is $(C \to \bot) \vee (C_1^\# \to \bot) \vee (C_2^\# \to \bot) \vee ...$, where $C, C_1, C_2, ...$ is an enumeration of $\mathfrak{B}(C)$. (Recall that $C^\#$ is just C.) Clearly, we should not expect to have $(C \to \bot) \vee (C_1^\# \to \bot) \vee (C_2^\# \to \bot) \vee ... \rhd C \to \bot$.

Line 2 of the argument will give us, in effect: $C \rhd (C \to \bot) \vee (C_1^\# \to \bot) \vee (C_2^\# \to \bot) \vee ....$ The obvious next step would be $1 \rhd (C \to (C \to \bot) \vee (C \to (C_1^\# \to \bot)) \vee (C \to (C_2^\# \to \bot)) \vee ...$, which is not intuitively valid.

It might be thought that some other intuitive form of reasoning that would do the job. Line 2 gives us something like: $C \rhd \exists B \in \mathfrak{B}(C)(B^\# \to \bot)$. With existential instantiation we would get that for some bunch, B, in $\mathfrak{B}(C)$, $C \rhd B^\# \to \bot$, so $C \otimes B \rhd \bot$. Hence for some B' in $\mathfrak{B}(C)$, $B' \rhd \bot$; and so, $1 \rhd Val(\langle C\rangle, \langle\bot\rangle)$. But existential instantiation will not work in this context. The move from $C \rhd \exists xD$ to $\exists x(C \rhd D)$ (x not free in C) is, again, obviously invalid.[17]

5.3 Fusion

But of course, all this presupposes that we can make sense of the machinery of bunches, and in particular of $\otimes$. This, as is clear, is simply a way of representing fusion, $\circ$. But what does fusion mean? $\circ$ is a sort of conjunction, since it is the residual of $\to$. That is, $A \rhd B \to C$ is equivalent to $A \circ B \rhd C$. But to get a better understanding of fusion, we need to turn to the semantics.

In the semantics, the truth conditions of $\circ$ are given in terms of the ternary relation (as are those of $\to$). Understanding the ternary relation is a vexed issue; but, it turns out, there are perfectly natural understandings of this.[18] In particular, we can think of $Rxyz$ as meaning that when functions in x are applied to the objects in y the results delivered are in z. We may then think of conditionals, semantically, as functions of a certain kind, from propositions to propositions. Specifically, $A \to B$ is a function which maps the proposition A to a proposition at least as strong as B. Thus, $\circ$ represents functional application. (And functional application is neither associative, commutative, nor contracting.)

From the perspective of this paper, then, the validity curry arises because certain logical machinery is too crude, forcing us to collapse important distinctions. Relevant logic has two sorts of conjunction, $\wedge$ and $\circ$. Classical logic collapses that distinction; but maintaining it solves the standard

[17] In connection with this, see Bacon (2013).

[18] See Beall, *et al.* (2012) and Priest (201+).

curry paradoxes. We do not have $((A \to B) \wedge A) \to B$, but we do have $((A \to B) \circ A) \to B$. What the validity curries appear to show us, perhaps unsurprisingly in retrospect, is that this distinction needs to be carried through in a more thorough-going manner: it needs to be applied not only for antecedents of conditionals but for premises of inferences. We need to distinguish between $\oplus$ and $\otimes$. We cannot get $X \rhd A \to B$ from $X \oplus A \rhd B$, but we can get it from $X \otimes A \rhd B$. Orthodox semantics—including those of standard relevant logics, which takes premises to be sets, or multisets—collapses this important distinction.

5.4 Two Concepts of Validity

I end on an historical note. In classical logic, because of *modus ponens* and conditional proof, we have:

$$\models A \to B \text{ iff } A \models B$$

The validity of a (one premise) argument can be defined in either way. Older logic texts used the left-hand definition; newer texts the right-hand one.

In relevant logic, with the usual semantics, the equivalence no longer holds. The left-hand side is stronger than the right. (It amounts to truth preservation over all worlds, rather than truth preservation over @.) So which notion gives the correct account of validity?[19]

The standard definition now is, perhaps, the right-hand one; but older texts in relevant logic, written before the semantics were well established, used the left-hand one.[20] And there are some good reasons to do so—quite independently of a solution to the validity curries. The right-hand approach divorces the conditional from entailment: $A \models B \to B$, but $\not\models A \to (B \to B)$; and $A \wedge (A \to B) \models B$, but $\not\models (A \wedge (A \to B)) \to B$. This is not necessarily an objection; but it does make the consequence relation look a bit odd from a relevant perspective, and it raises the question of what, exactly, the conditional means.[21]

[19]The distinction is analogous to one found in discussions of the logic of the *actually* operator. See Humberstone (2004). The best account there is also moot. See Hanson (2006).

[20]I owe this observation to Stephen Read.

[21]From the present perspective, what happens to the validity curry argument with this notion of validity? The answer is that $V1$ breaks down, since truth preservation at the base world does not guarantee truth preservation at all worlds. A slightly different version of the argument appeals to the principle $V3$: $V(\langle C \rangle, \langle D \rangle) \oplus C \rhd D$. Lines 2 and 4 of (**) give us that $C \rhd V(\langle C \rangle, \langle \perp \rangle) \oplus \perp$, and Cut then gives us $C \rhd \perp$. But there is no reason to suppose that $V3$ holds. What does hold is the same thing with $\otimes$ instead of $\oplus$; and

The left-hand approach, by contrast, looks more natural. Indeed, one of the motivating thoughts of early relevant logic was exactly to have a connective in the language which expressed entailment.[22] Thus, validity is expressed by the logical truth of the conditional, as a matter of definition. And an inference from A to B is valid iff A suffices logically for B; that is, in every interpretation, and any world of that interpretation, w, if A is true at w, so is B. All these pieces of the jigsaw then fit together nicely.

In some sense, then, this paper represents a return to the roots of relevant logic.[23]

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with this principle, the argument will not go through without an illicit contraction. Being valid is a sort of conditional; and conditionals require intentional connection to produce the consequent.

[22] This is very clear in Anderson and Belnap (1970). See pp. 1-29 and (esp.) pp. 473-92.

[23] A version of this paper was given at the workshop on Curry's Paradox at the University of Otago, and to a meeting of the Melbourne Logic group both in August 2012. Thanks go to the participants for their helpful thoughts and comments, and especially to Greg Restall, Dave Ripley, and Jerry Seligman. A particular thanks goes to Lloyd Humberstone, both for comments at the meeting and for substantial written comments. Thanks also go to two anonymous referees for this volume.

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